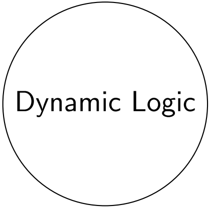
Dynamic Logic

How loops are actually recursion
Hype for Types

Jacob Neumann

08 October 2019

Dynamic Logic is everywhere



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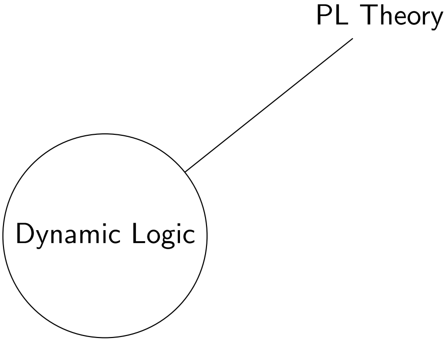

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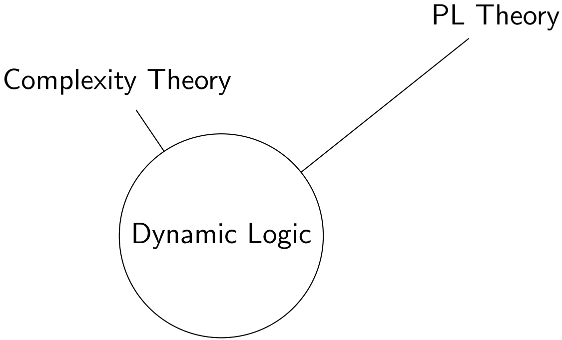

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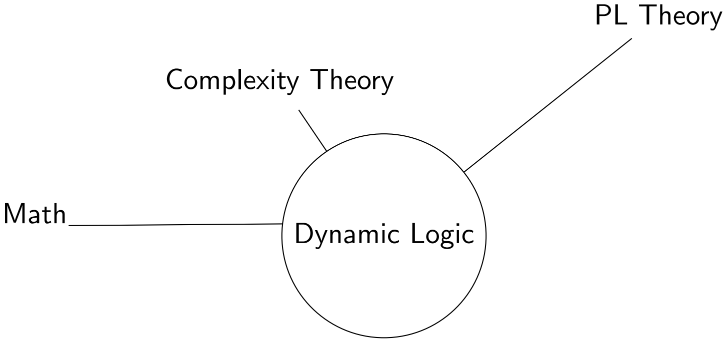

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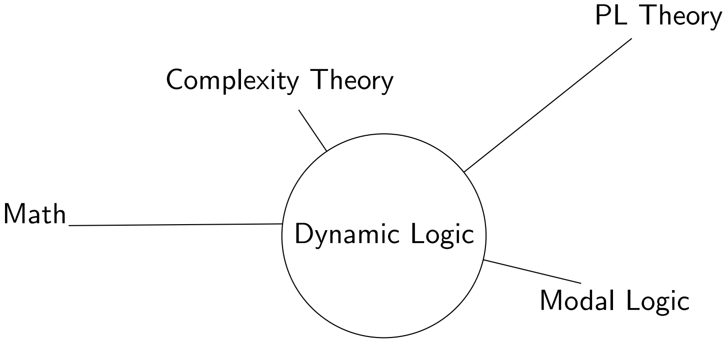

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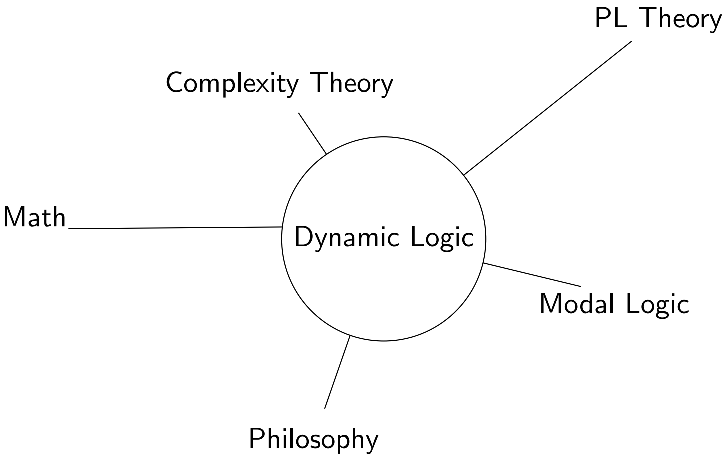

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WARNING

IMPERATIVE CODE AHEAD

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(also math)

Section 1

Syntax and Semantics

Some philosophy...

Question: What is programming?

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(One possible) answer:

- Programming is the art of communicating with computers

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(One possible) answer:

- Programming is the art of communicating with computers
- We communicate with computers using otherwise-meaningless strings of symbols

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while true:  print("AHHHH")
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fun fact 0 = 1
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```
/([a-z0-9\.-]+)@([\da-z\.-]+)\.([a-z\.]{2,6})/
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- Be sure our code will return the right result
- Know how long our code will take to run
- Be sure that we won't run into unforeseen bugs at runtime

Operational vs. Denotational

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In this lecture, I'll be focusing on the *denotational* approach.

Section 2

Deterministic PDL

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- A state space
- Interpretations of all the programs as *partial functions* on the state space
- A apparatus for formulating logical statements about the state space

The State Space

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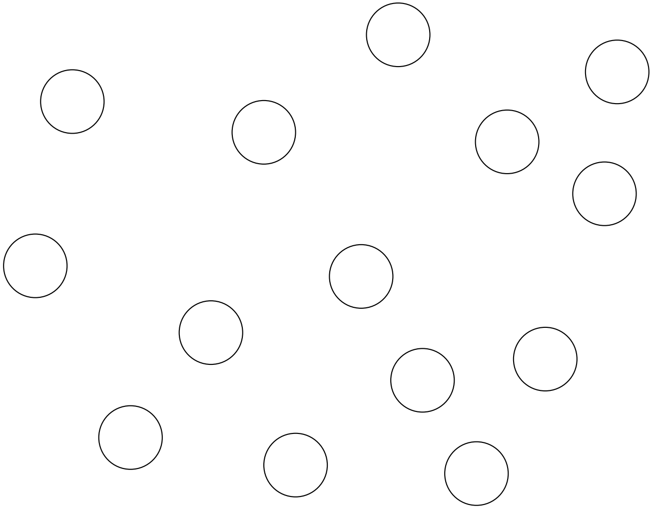
X is often either finite or countably infinite, although in some applications we will want to have more states.

The State Space



The Programs

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Each program symbol $\pi \in \Pi$ *denotes* a partial function on our state space. We write this as:

$$\|\pi\| : X \rightharpoonup X$$

So, for each state $x \in X$, "executing π at x" will either succeed (terminate) and result in a new state $\|\pi\|(x)$, or it will crash (encoded by $\|\pi\|(x)$ being undefined).

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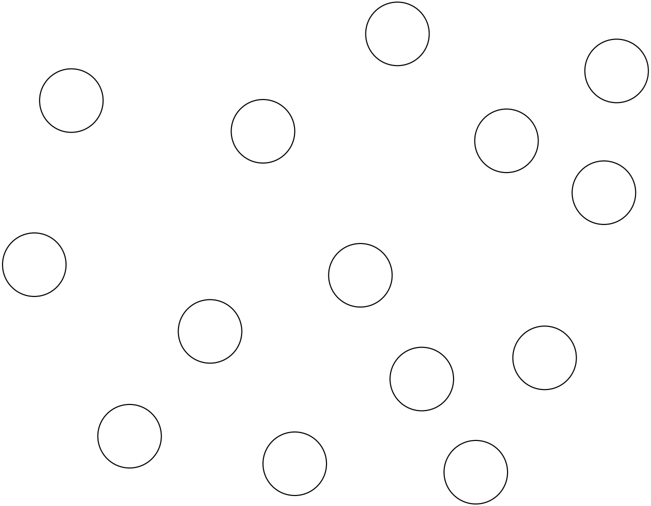

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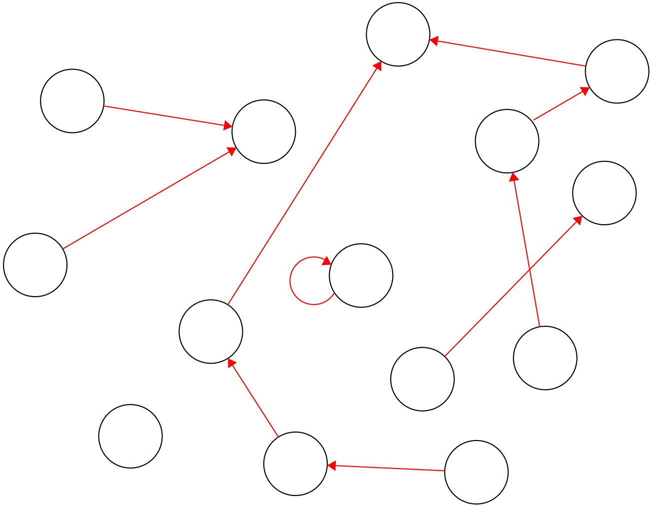
π_0



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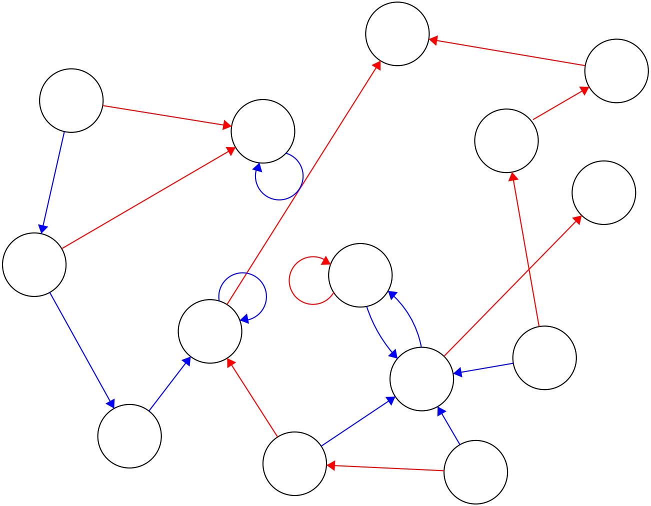

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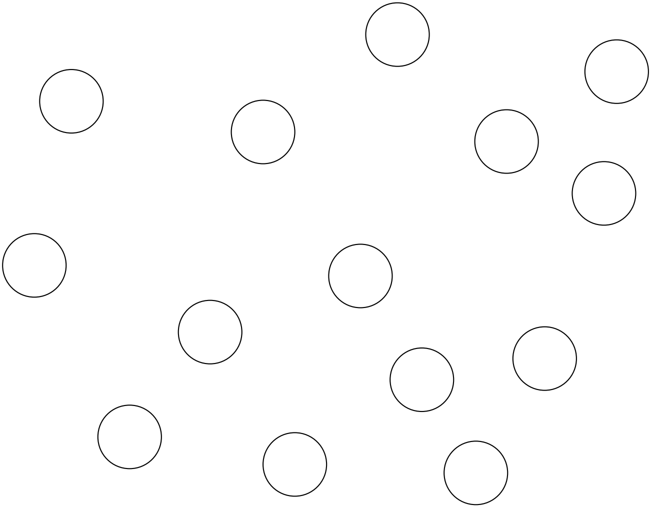
Each propositional variable $p \in \Phi$ *denotes* a subset of our state space. We write this as:

$$[\![p]\!] \subseteq X$$

Think of $[\![p]\!]$ as the set of states where p is true.

The Propositions



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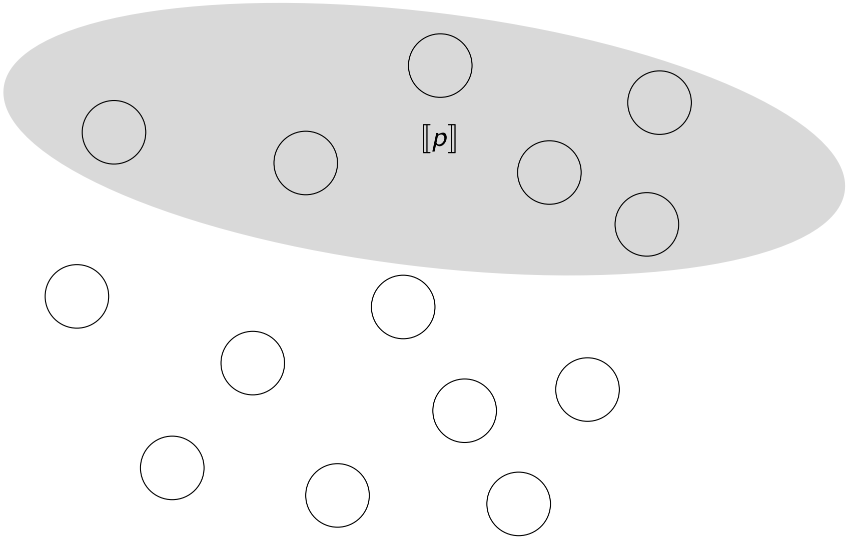

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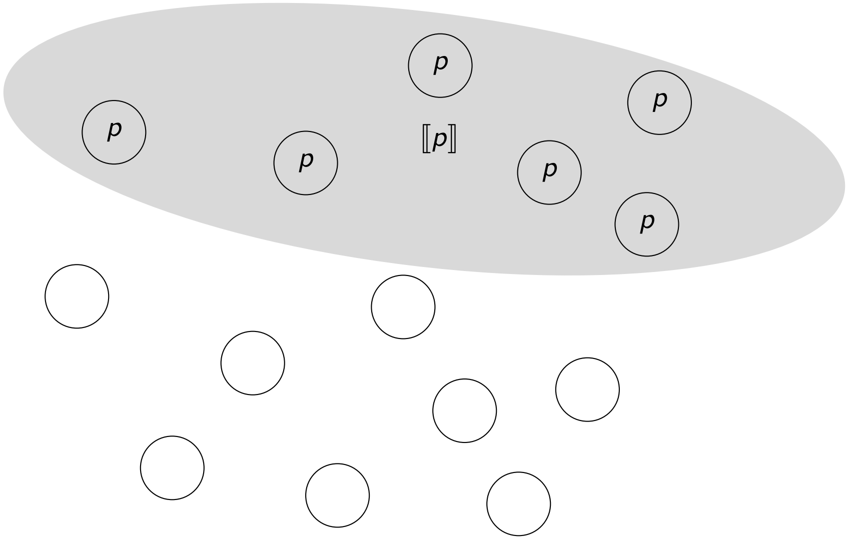

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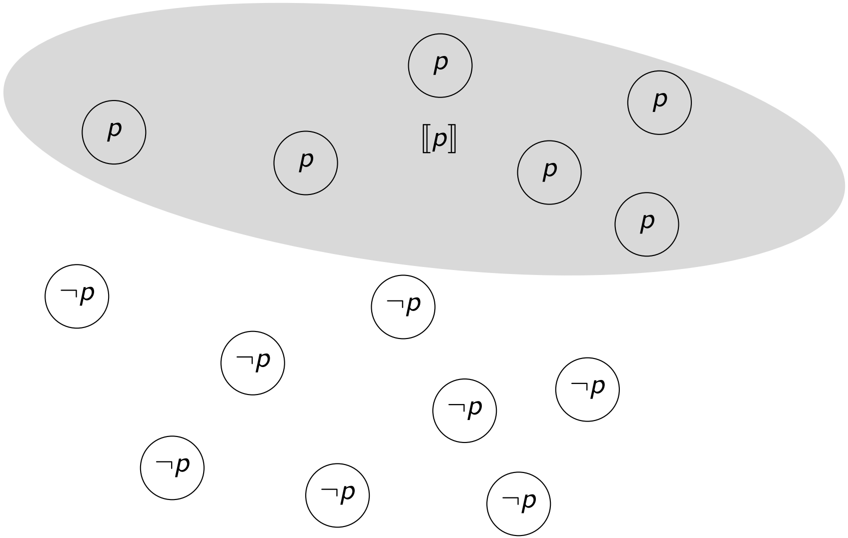

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The formula $\langle\pi\rangle\varphi$, which is defined to be $\neg[\pi]\neg\varphi$, expresses the statement "π terminates, and results in a φ state".

Sanity Check: What does this formula say?

$$\varphi \rightarrow \langle \pi \rangle \psi$$

(here, $p \rightarrow q$ is used as an abbreviation for $\neg p \vee q$).

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REQUIRES: φ
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So we have given semantics for a simple programming language, with:

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- Whatever basic programs we might want
- Sequencing, conditionals, and loops
- A logical syntax to talk about state properties before and after executing a function

Section 3

Proving Behavior in DPDL

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- $$\frac{(\varphi \wedge \psi) \to [\pi]\psi}{\psi \to [\textbf{while } \varphi \textbf{ do } \pi](\neg\varphi \wedge \psi)}$$

Section 4

Hoare Logic

From PDL to Hoare Logic

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- $\{\mathtt{n} \geq 0\}\, i := n \,\{\mathtt{i} \geq 0\}$
- $\left\{\mathtt{a} = \mathtt{b}^{\mathtt{i}}\right\} a := a * b \left\{\mathtt{a} = \mathtt{b}^{\mathtt{i+1}}\right\}$

Here are our rules from earlier, in the Hoare notation:

- $$\frac{\{\varphi\}\,\pi_1\,\{\psi\} \qquad \{\psi\}\,\pi_2\,\{\theta\}}{\{\varphi\}\,\pi_1;\pi_2\,\{\theta\}}$$

- $$\frac{\{\varphi\}\,\pi_1\,\{\psi\} \qquad \{\neg\varphi\}\,\pi_2\,\{\psi\}}{\{\}\,\textbf{if }\varphi\textbf{ then }\pi_1\textbf{ else }\pi_2\,\{\psi\}}$$

- $$\frac{\{\varphi \wedge \psi\}\,\pi\,\{\psi\}}{\{\psi\}\,\textbf{while }\varphi\textbf{ do }\pi\,\{\neg\varphi \wedge \psi\}}$$

A 122-style example

```
i:=n;
res:=1;
(while (i>0)
do

   res := res * b;
   i := i-1

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i:=n;
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   res := res * b;
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$\{\mathtt{n} \geq 0\}$

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- Nondeterministic program semantics (replace the partial functions with binary relations)
- Heap Allocation
- Concurrency
- Cost Semantics
- More complex mathematics to make the modal logic more powerful (topological structure, structure-preserving maps and category theory, etc.)

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  - This approach can be extended to type theories which are *much* fancier than simple lambda calculus (e.g. dependent type theory, higher-inductive dependent type theory (homotopy type theory)). This is an area of very active research.

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  - Introduce the mathematical language of *category theory*
  - Develop denotational semantics for the simply-typed lambda calculus *inside certain nice categories*
  - This approach can be extended to type theories which are *much* fancier than simple lambda calculus (e.g. dependent type theory, higher-inductive dependent type theory (homotopy type theory)). This is an area of very active research.
- So stay tuned...

Thank you!